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1 Theoretical Study of The Generalized Laguerre Polynomials

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Abstract

The Laguerre polynomials play an important role in several diverse fields of physics and in many given physical problems a numerical analytical evaluation of the Laguerre polynomials is required.

This paper aims at presenting Laguerre polynomials in two variables which provide an unification and a generalization of the classical Laguerre polynomials and their various generalizations introduced in the literatures from time to time. In particular, generating functions, integral representations, differential equation, recurrence relations, Laplace transform, expansions and some applications of this polynomials are established. We also derive explicit representation for this Laguerre polynomials defined through generalized hypergeometric functions, which naturally yield numerous other potentially useful (numerical and analytical) properties of the Laguerre polynomials. Finally, some of these results are employed to derived explicit relationships for the wavefunctions of the Morse and Pöschl-Teller potentials, which are frequently used to describe molecular systems.

Keywords: mathematical physics, Laguerre polynomials, generating relations, generalized hypergeometric functions, wavefunctions.

الملخص

يلعب التشكيل الحسابي للأجير دورهام في عدة حقول مختلفة للفيزياء. كما أن التحليل العددي للتشكيل الحسابي للأجير مطلوب في كثير من المسائل الفيزيائية. هذا البحث يهدف إلى تقديم تشكيل حسابي للأجير ذات متغيرين كتحوير وتطوير للتشكيلات الحسابية للأجير البدائية وكذلك للعديد من الأنواع المطورة لها التي يتم تقديمها في المراجع العلمية بين الحين والأخر. وسوف نناقش في هذا البحث عدة صفات للتشكيل الحسابي للأجير ذات المتغيرين والتي تتلخص في إنشاء دوال مولدة لها , تمثيلات تكاملية, معادلة تفاضلية, صيغ معاودة, تحويل لابلاس , توسعات حدية وبعض التطبيقات. بالإضافة إلى ذلك تم اشتقاق تمثيل صريح للتشكيل الحسابي للأجير معرف من خلال دوال فوق هندسية مطورة, والتي من الطبيعي بدورها تنتج العديد من الصفات الأساسية المفيدة في التحليل العددي للتشكيل الحسابي للأجير. في الأخير بعض هذه النتائج وظفت لاشتقاق علاقات صريحة لدوال الموجات التابعة لقوى مورس و بوشلر والتي تستعمل بشكل متكرر في وصف الأنظمة الجزئية.

1. INTRODUCTION

The Laguerre polynomials $L_n(x)$ and the associated Laguerre polynomials $L_n^{(\alpha)}(x)$ define by the series

$$L_n(x) = \sum_{s=0}^{n} \binom{n}{n-s} \frac{(-x)^s}{s!} \quad , \tag{1}$$

$$L_n^{(\alpha)}(x) = \sum_{s=0}^{n} \binom{\alpha+n}{n-s} \frac{(-x)^s}{s!} \quad , \tag{2}$$

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are important classes of orthogonal polynomials encountered in the applications, especially in problems involving the integration of Helmholtz's equation in the parabolic coordinates, in the theory of the hydrogen atom, in the theory of propagation of electrometric waves along transmission lines ,etc. For an excellent review of various mathematical properties and computational methods concerning the Laguerre polynomials [Lebedev, 1965; Erdelyi, 1953; and Anderws, 1985].
Also, the theory of the generalized Laguerre polynomials has witnessed a rather significant evolution during the last years ([Dattoli et. al., 1996, 1998*a* and 2000*b*]). In applicative fields, we note that for some physical problems the use of new classes of polynomials provided solutions hardly achievable with conventional analytical and numerical means. In this paper we aim at establishing a new class of Laguerre polynomials denoted by $L_n^{(\alpha,\beta)}(x,y;k)$ involving two variables , which are generalization and unification of number of known Laguerre polynomials defined by Ragab, 1991], Dattoli.et.al., 1998*a* and Knohauser, 1965] and obviously the ordinary Laguerre polynomials defined by(1) and (2). Further,a number of properties of these polynomials are discussed, including, generating relations, differential equation, differential and recurrence relations, integral representations, Laplace transform, series expansion and some of applications.

2. THE LAGUERRE POLYNOMIALS $L_n^{(\alpha,\beta)}(x,y;k)$

We defined the generalized Laguerre polynomials $L_n^{(\alpha,\beta)}(x,y;k)$ by means of the generating relation:

$$e^t\,{}_0F_1(-;\alpha+1;-xt)\;{}_0F_k\left[-;\frac{\beta+1}{k},\ldots,\frac{\beta+k}{k};-(y/k)^k t\right]$$
$$=\sum_{n=0}^{\infty}\frac{n!\,L_n^{(\alpha,\beta)}(x,y;k)}{(\beta+1)_{kn}(\alpha+1)_n}\,t^n \quad , \qquad (3)$$

where ${}_pF_q$ is the generalized hypergeometric function[Srivastava and Manoch,1984]:

$${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!} \quad , \qquad (4)$$

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma\lambda} \quad , \qquad \Gamma: \textit{Gamma function} \quad ,$$

$$(\lambda)_{kn}=k^{kn}\left(\frac{\lambda}{k}\right)_n\left(\frac{\lambda+1}{k}\right)_n\cdots\left(\frac{\lambda+k-1}{k}\right)_n \; , \; k=1,2,3,\ldots \textit{ and } n=0,1,2,\ldots \quad .$$

From (3) and (4), we get

$$\sum_{n=0}^{\infty}\frac{n!\,L_n^{(\alpha,\beta)}(x,y;k)}{(\beta+1)_{kn}(\alpha+1)_n}\;t^n=\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\sum_{q=0}^{\infty}\frac{(-x)^q}{q!(\alpha+1)_q}\sum_{s=0}^{\infty}\frac{(-y^k)^s}{s!(\beta+1)_{ks}}\right\}\;t^{n+q+s} \quad ,$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{s=0}^{n}\sum_{q=0}^{n-s}\frac{(-y^k)^s(-x)^q}{s!q!(\beta+1)_{ks}(\alpha+1)_q(n-s-q)!}\right\}\;t^n \; .$$

which on using $(n-k)!=\frac{(-1)^k n!}{(-n)_k}$ and comparing the coefficients of t^n, yields

$$L_n^{(\alpha,\beta)}(x,y;k)=\frac{\Gamma(kn+\beta+1)\Gamma(\alpha+n+1)}{(n!)^2}\sum_{s=0}^{n}\sum_{q=0}^{n-s}\frac{(-n)_{q+s}\;x^q\;y^{ks}}{\Gamma(ks+\beta+1)\Gamma(\alpha+q+1)q!s!}. \qquad (5)$$

equation (5) shows that $L_n^{(\alpha,\beta)}(x,y;k)$ is polynomials in y^k and x.

It may be of interest to point out that (5) yields the following relationships:

$$L_n^{(0,0)}(x,0;1)=L_n(x) \quad , \qquad (6)$$
$$L_n^{(\alpha,0)}(x,0;1)=L_n^{(\alpha)}(x) \quad , \qquad (7)$$
$$L_n^{(0,\beta)}(0,y;k)=Z_n^{\beta}(y;k) \quad , \qquad (8)$$
$$y^n L_n^{(0,0)}(x/y,0;1)=L_n(x,y) \quad , \qquad (9)$$
$$y^n L_n^{(m,0)}(x/y,0;1)=L_n^{(m)}(x,y) \quad , \qquad (10)$$
$$\frac{n!\;x^{\rho}\;y^n}{(n+\rho)!}L_n^{(\rho,0)}(-x/y,0;1)={}_1L_{n,\rho}(x,y) \quad , \qquad (11)$$

and

$$L_n^{(\alpha,\beta)}(x,y;1)=L_n^{(\alpha,\beta)}(x,y) \quad , \qquad (12)$$

are Laguerre polynomials due to Ragab[1991] defined by: $L_n^{(\alpha,\beta)}(x,y)$ where

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{s=0}^{n}\sum_{q=0}^{n-s}\frac{(-n)_{s+q}}{(\alpha+1)_s(\beta+1)_q\,s!q!}\frac{y^s}{s!}\frac{x^q}{q!} \quad , \qquad (13)$$

$Z_n^{\alpha}(x;k)$ is Konhouser polynomial [1965]:

$$Z_n^{\alpha}(x;k)=\frac{\Gamma(kn+\alpha+1)}{n!}\sum_{s=0}^{n}\frac{(-n)_s}{\Gamma(ks+\alpha+1)}\frac{x^{ks}}{s!} \quad , \qquad (14)$$

k is positive integer,which for *k=1* reduces to the associated Laguerre polynomials (2) and the special case when *k=2* were encountered earlier by Spencer and Fano [1951] in certain calculation involving the penetration of gamma rays through matter, and $L_n(x,y)$, ${}_1L_{n,\rho}(x,y)$ and $L_n^m(x,y)$ are Laguerre polynomials due to [Dattoli et.al., 1998*a*; 1999 and 2000*b*].

Clearly, for $y=\beta=0$ and $k=1$,(see (7)), (3) immediately reduces to the well known generating relation of the associated Laguerre polynomials $L_n^{(\alpha)}(x)$ [Rainville, 1960]:

$$e^t\,{}_0F_1(-;\alpha+1;-xt)=\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n}t^n \quad . \qquad (15)$$

Whereas, for $x=\alpha=0$, (see (8)), equation(3) reduces to another known result due to [Srivastava, 1982] in the form:

$$\sum_{n=0}^{\infty}Z_n^{\beta}(y;k)\frac{t^n}{(\beta+1)_{kn}}=e^t\,{}_0F_k\left[-;\frac{\beta+1}{k},\ldots,\frac{\beta+1}{k};-(y/k)^k t\right] \quad . \qquad (16)$$

Now ,one more particular case of the generating function (3) is worthily of note. Indeed, if in (3) , we set $y=\alpha=\beta=0, k=1$, replace t by ty and x by x/y and use the relationship (9), we obtain:

$$e^{yt} C_0(xt)=\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x,y) \qquad , \tag{17}$$

where $C_0(x)$ is Tricomi Bessel function [Dattoli and Torre, 1996]:

$$C_0(x)=\sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m!)^2} \qquad . \tag{18}$$

In fact the above formula (17) is a known result (see [Dattoli and et.al. 2000*b*]).
Further in view of (7) and (8), we may write (5) in the more elegant forms:

$$L_n^{(\alpha,\beta)}(x,y;k)=\frac{\Gamma(kn+\beta+1)\Gamma(\alpha+n+1)}{n!} \sum_{s=0}^{n} \frac{(-y^k)^s L_{n-s}^{(\alpha)}(x)}{s!\Gamma(\alpha+s+1)\Gamma[k(n-s)+\beta+1]} \quad , \tag{19}$$

and

$$L_n^{(\alpha,\beta)}(x,y;k)=\frac{\Gamma(kn+\beta+1)\Gamma(\alpha+n+1)}{n!} \sum_{s=0}^{n} \frac{(-x)^s Z_{n-s}^{\beta}(y;k)}{s!\Gamma(\alpha+s+1)\Gamma[k(n-s)+\beta+1]} \quad , \tag{20}$$

respectively.
Furthermore, according to the definition of the Kampé de Fériet series of two variables $F_{l:m;n}^{p:q;k}$ [Srivastava and Manocha,1984]:

$$F_{l:m;n}^{p:q;k}\left[\begin{matrix}(a_p):(b_q);(c_k);\\ (d_l):(e_m);(f_n);\end{matrix} x,y\right]=\sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(d_j)_{r+s}\prod_{j=1}^{m}(e_j)_r\prod_{j=1}^{n}(f_j)_s} \frac{x^r}{r!}\frac{y^s}{s!} \quad , \tag{21}$$

we can easily derive the following explicit representation of $L_n^{(\alpha,\beta)}(x,y;k)$:

$$L_n^{(\alpha,\beta)}(x,y;k)=\frac{(\beta+1)_{kn}(\alpha+1)_n}{(n!)^2} F_{0:1;k}^{1:0;0}\left[\begin{matrix}-n:\text{——};\text{——};\\ -:\alpha+1;\Delta(k,\beta+1);\end{matrix} x,(\frac{y}{k})^k\right] \quad , \tag{22}$$

where $\Delta(k;\lambda)$ denotes the array of k parameters : $\frac{\lambda}{k}$, $\frac{\lambda+1}{k}$,..., $\frac{\lambda+k-1}{k}$.

Yet, another generating function for $L_n^{(\alpha,\beta)}(x,y;k)$ can be established in the form:

$$\sum_{n=0}^{\infty} \frac{n!(\lambda)_n L_n^{(\alpha,\beta)}(x,y;k)}{(\beta+1)_{kn}(\alpha+1)_n} t^n=(1-t)^{-\lambda} F_{0:1;k}^{1:0;0}\left[\begin{matrix}\lambda:\text{——};\text{——};\\ -:\alpha+1;\Delta(k,\beta+1);\end{matrix}\left(\frac{-xt}{1-t}\right)\left[\frac{-y^k t}{(1-t)k^k}\right]\right], \tag{23}$$

To establish this generating function, we have from (5):

$$\sum_{n=0}^{\infty}\frac{n!(\lambda)_n L_n^{(\alpha,\beta)}(x,y;k)}{(\beta+1)_{kn}(\alpha+1)_n}t^n=\sum_{n=0}^{\infty}\frac{(\lambda)_n\Gamma(\beta+1)\Gamma(\alpha+1)}{n!}\sum_{s=0}^{n}\sum_{m=0}^{n-s}\frac{(-n)_{m+s}\; y^{ks}\, x^m\, t^n}{\Gamma(ks+\beta+1)\Gamma(\alpha+m+1)s!m!}$$

$$\sum_{s=0}^{n}\sum_{m=0}^{n-s}\frac{(\lambda)_{m+s}(-tx)^m(-ty^k)^s}{(\beta+1)_{ks}(\alpha+1)_m m!s!}\sum_{n=0}^{\infty}\frac{(\lambda+m+s)_n}{n!}t^n =$$

$$.=(1-t)^{-\lambda}F^{1:0;0}_{0:1;k}\left[\begin{matrix}\lambda:\text{———};\text{————};\\ -:\alpha+1;\Delta(k,\beta+1);\end{matrix}\left(\frac{-xt}{1-t}\right)\left(\frac{-y^k t}{(1-t)k^k}\right)\right]$$

For $x=\alpha=0$ and in view of (8), (23) reduces to a known generating relation due to [Prabhakar, 1970]:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n Z_n^{\beta}(y;k)}{(\beta+1)_{kn}}t^n=(1-t)^{-\lambda}\,{}_1F_k\left[\lambda;\Delta(k,\beta+1);\left(\frac{-y^k t}{(1-t)k^k}\right)\right]\quad . \tag{24}$$

3. INTEGRAL REPRESENTATIONS FOR $L_n^{(\alpha,\beta)}(x,y;k)$

We have the following integral representation for $L_n^{(\alpha,\beta)}(x,y;k)$:

$$L_n^{(\alpha,\beta)}(x,y;k)=\frac{y^{-\beta}\Gamma(kn+\beta+1)}{\Gamma(\beta-\mu)\Gamma(kn+\mu+1)}\int_0^y t^{\mu}(y-t)^{\beta-\mu-1}L_n^{(\alpha,\mu)}(x,t;k)\;dt \quad , \tag{25}$$

$$L_n^{(\alpha,\beta)}(x,y;k)=\frac{x^{-\alpha}\Gamma(n+\alpha+1)}{\Gamma(\alpha-\mu)\Gamma(n+\mu+1)}\int_0^x t^{\mu}(x-t)^{\alpha-\mu-1}L_n^{(\mu,\beta)}(t,y;k)\;dt \quad , \tag{26}$$

for Re(β)>Re(μ)>-1 and Re(α)>Re(μ)>-1 ,

and

$$L_n^{(\alpha,\beta)}(x,y;k)=\frac{\Gamma(kn+\beta+1)\Gamma(\alpha+n+1)}{(n!)^2(2\pi i)^2}\int_{-\infty}^{(0+)}\int_{-\infty}^{(0+)}\frac{[t^k-(xt^k/\mu)-y^k]^n}{t^{kn+\beta+1}\mu^{\alpha+1}}\;e^{t+\mu}dt\,d\mu . \tag{27}$$

To establish (25) , substitute for $L_n^{(\alpha,\beta)}(x,y;k)$ from (5), using the integral formula [Srivastava and Manocha, 1984]:

$$\int_0^x t^{\alpha-1}(x-t)^{\beta-1}dt=\frac{\Gamma\alpha\;\Gamma\beta}{\Gamma(\alpha+\beta)}\;x^{\alpha+\beta-1} \quad , \tag{28}$$

and finally (25) follows from (5). In the same way (26) can be proved.

Finally, upon using the binomial theorem:

$$(1-x-y)^{-\lambda}=\sum_{s,m=0}^{\infty}(\lambda)_{s+m}\frac{x^s}{s!}\frac{y^m}{m!} \quad , \tag{29}$$

and Hankel's formula [Erdélyi,1953]:

$$\frac{1}{\Gamma z}=\frac{1}{2\pi i}\int_{-\infty}^{(0+)}e^t t^{-z}dt \quad , \tag{30}$$

we can establish (27).

For $x = \alpha = 0$, (25) reduces to the known result [Prabhakar and Rekha,1978]:

$$Z_n^{\beta}(y;k) = \frac{y^{-\beta}\Gamma(kn+\beta+1)}{\Gamma(\beta-\mu)\Gamma(kn+\mu+1)} \int_0^y t^{\mu}(y-t)^{\beta-\mu-1} Z_n^{\mu}(t;k)\ dt\,, \tag{31}$$

If in (25) and (26) we let *k=1*, we get interesting integral representations for the Laguerre polynomials due to Ragab $L_n^{(\alpha,\beta)}(x,y)$ [1991] as follows:

$$L_n^{(\alpha,\beta)}(x,y) = \frac{y^{-\beta}\Gamma(\beta+n+1)}{\Gamma(\beta-\mu)\Gamma(n+\mu+1)} \int_0^y t^{\mu}(y-t)^{\beta-\mu-1} L_n^{(\alpha,\mu)}(x,t)dt \quad , \tag{32}$$

and

$$L_n^{(\alpha,\beta)}(x,y) = \frac{x^{-\alpha}\Gamma(\alpha+n+1)}{\Gamma(\alpha-\mu)\Gamma(n+\mu+1)} \int_0^x t^{\mu}(x-t)^{\alpha-\mu-1} L_n^{(\mu,\beta)}(t,y)dt \quad . \tag{33}$$

respectively.

4. RECURRENCE RELATIONS AND DIFFERENTIAL EQUATION

Consider the expression:

$$I = (\alpha+n)L_n^{(\alpha-1,\beta)}(x,y;k) + \frac{x\Gamma(kn+\beta+1)}{n(kn-k+\beta+1)} L_{n-1}^{(\alpha+1,\beta)}(x,y;k) \quad . \tag{34}$$

Substituting for $L_n^{(\alpha,\beta)}(x,y;k)$ from (5), we get

$$I = \frac{\Gamma(kn+\beta+1)\Gamma(\alpha+n+1)}{(n!)^2} \sum_{s=0}^{n}\sum_{m=0}^{n-s} \frac{(-n)_{m+s}\, x^m y^s}{\Gamma(ks+\beta+1)\Gamma(m+\alpha)m!s!} + \frac{x\Gamma(kn+\beta+1)\Gamma(\alpha+n+1)}{n[(n-1)!]^2} \times \sum_{s=0}^{n}\sum_{m=0}^{n-s} \frac{(-n+1)_{m+s}\, x^m y^s}{\Gamma(ks+\beta+1)\Gamma(\alpha+m+2)m!s!} \quad . \tag{35}$$

Now, on letting $m = m-1$ in the second term of (35),and simplifying, we obtain the recurrence relation:

$$\alpha\, L_n^{(\alpha,\beta)}(x,y;k) = (\alpha+n)L_n^{(\alpha-1,\beta)}(x,y;k) + \frac{x\,\Gamma(kn+\beta+1)}{n\Gamma(kn-k+\beta+1)} L_{n-1}^{(\alpha+1,\beta)}(x,y;k) \ . \tag{36}$$

In the same manner ,we can show that

$$\beta L_n^{(\alpha,\beta)}(x,y;k) = (kn+\beta)L_n^{(\alpha,\beta-1)}(x,y;k) + \frac{ky(\alpha+n)}{n} L_{n-1}^{(\alpha,\beta+k)}(x,y;k) \ . \tag{37}$$

For $k=1$ and using the relationship (12) , equations (36) and (37) reduce to a known result due to [Ragab, 1991] :

$$\alpha\, L_n^{(\alpha,\beta)}(x,y) = (\alpha+n)L_n^{(\alpha-1,\beta)}(x,y) + \frac{x\Gamma(n+\beta+1)}{n\Gamma(n+\beta)} L_{n-1}^{(\alpha+1,\beta)}(x,y) \quad , \tag{38}$$

and

$$\beta L_n^{(\alpha,\beta)}(x,y) = (n+\beta)L_n^{(\alpha,\beta-1)}(x,y) + \frac{y(\alpha+n)}{n} L_{n-1}^{(\alpha,\beta+1)}(x,y) \quad , \tag{39}$$

respectively.

In view of the operator D_z^n define by (see[Dattoli and Torre,1998*a*]):

$$D_z^n z^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} z^{\alpha-n} \quad , \tag{40}$$

we can easily see that:

$$D_x\, L_n^{(\alpha,\beta)}(x,y;k) = \frac{-\Gamma(kn+\beta+1)}{n\Gamma(kn-k+\beta+1)} L_{n-1}^{(\alpha+1,\beta)}(x,y;k) \quad , \tag{41}$$

$$y^{1-k} D_y\, L_n^{(\alpha,\beta)}(x,y;k) = \frac{-k(n+\alpha)}{n} L_{n-1}^{(\alpha,\beta+k)}(x,y;k) \quad , \tag{42}$$

$$D_x x^{\alpha+1} D_x\, L_n^{(\alpha,\beta)}(x,y;k) = \frac{-x^\alpha (n+\alpha)\Gamma(kn+\beta+1)}{n\Gamma(kn-k+\beta+1)} L_{n-1}^{(\alpha,\beta)}(x,y;k) \quad , \tag{43}$$

$$D_y^k y^{\beta+1} D_y\, L_n^{(\alpha,\beta)}(x,y;k) = \frac{-ky^\beta (n+\alpha)\Gamma(kn+\beta+1)}{n\Gamma(kn-k+\beta+1)} L_{n-1}^{(\alpha,\beta)}(x,y;k) \quad , \tag{44}$$

$$D_x^{-1} x^{\alpha-1} D_x^{-1}\, L_n^{(\alpha,\beta)}(x,y;k) = \frac{-(n+1)x^\alpha}{(kn+\beta+1)_k(\alpha+n+1)} L_{n+1}^{(\alpha,\beta)}(x,y;k) \quad , \tag{45}$$

and

$$D_y^{-k} y^{\beta-k} D_y^{-1}\, L_n^{(\alpha,\beta)}(x,y;k) y^{k-1} = \frac{-(n+1)y^\beta}{k(kn+\beta+1)_k(\alpha+n+1)} L_{n+1}^{(\alpha,\beta)}(x,y;k) . \tag{46}$$

Next, from the relations (43) and (44), it is also proved that the set of polynomials $L_n^{(\alpha,\beta)}(x,y;k)$ satisfy the partial differential equation:

$$x^{-\alpha} \frac{\partial}{\partial x}\left(x^{\alpha+1} \frac{\partial}{\partial x} L_n^{(\alpha,\beta)}(x,y;k) \right) = \frac{y^{-\beta}}{k} \frac{\partial^k}{\partial y}\left(y^{\beta+1} \frac{\partial}{\partial y} L_n^{(\alpha,\beta)}(x,y;k) \right) \quad . \tag{47}$$

Now, in (3) denoting $e^t\, {}_0F_1(-;\alpha+1;-xt)\, {}_0F_k\left[-; \frac{\beta+1}{k}, \ldots, \frac{\beta+k}{k}; -\left(\frac{y}{k}\right)^k t \right]$ by $f(x,y;t)$, we at once find that $f(x,y;t)$ satisfies the partial differential equation:

$$x\frac{\partial f}{\partial x} + \frac{y}{k}\frac{\partial f}{\partial y} - t\frac{\partial f}{\partial t} + t\, f = 0 \quad . \tag{48}$$

Substituting for $f(x,y;t)$ from (3) and equating the coefficients of t^n, we obtain the differential recurrence relation:

$$xD_x\, L_n^{(\alpha,\beta)}(x,y;k) + \frac{y}{k} D_y\, L_n^{(\alpha,\beta)}(x,y;k)$$
$$= nL_n^{(\alpha,\beta)}(x,y;k) - \frac{\Gamma(kn+\beta+1)}{n(n+\alpha)\Gamma(kn-k+\beta+1)} L_n^{(\alpha,\beta)}(x,y;k) . \tag{49}$$

5. LAPLACE TRANSFORM

Operational image of the Laguerre polynomials $L_n^{(\alpha,\beta)}(x,y;k)$ in the classical Laplace transform [Srivastava and Manocha, 1984]:

$$L\{f(t):s\} = \int_0^\infty e^{-st} f(t)dt = F(s) \quad , \tag{50}$$

can be obtained by appealing to Euler's integral :

$$\int_0^\infty t^{\lambda-1}e^{-st}dt = \frac{\Gamma\lambda}{s^\lambda} \quad ,\min\{\mathrm{Re}(\lambda),\mathrm{Re}(s)\} > 0 \quad . \tag{51}$$

Thus ,we have

$$\int_0^\infty\int_0^\infty u^\gamma v^\lambda e^{-s_1u-s_2v}\, L_n^{(\alpha,\beta)}(xu,yv;k)\,du\,dv$$

$$= \frac{(\beta+1)_{kn}(\alpha+1)_n\Gamma(\gamma+1)\Gamma(\lambda+1)}{(n!)^2 s_1^{\gamma+1}s_2^{\lambda+1}} F^{1:1;k}_{0:1;k}\left[\begin{matrix} -n:\gamma+1;\Delta(k,\lambda+1); \\ -:\alpha+1;\Delta(k,\beta+1); \end{matrix} \frac{x}{s_1},\left(\frac{y}{s_2}\right)^k\right], \tag{52}$$

provided that $\mathrm{Re}(s_1) > 0, \mathrm{Re}(s_2) > 0, \mathrm{Re}(\gamma) > -1$ *and* $\mathrm{Re}(\lambda) > -1$.

Now, some special cases of (52) are worthy to be noted. For $k=1$ and using the relationship (12), (52) reduces to

$$\int_0^\infty\int_0^\infty u^\gamma v^\lambda e^{-s_1u-s_2v}\, L_n^{(\alpha,\beta)}(xu,yv)\,du\,dv$$

$$= \frac{(\beta+1)_{kn}(\alpha+1)_n\Gamma(\gamma+1)\Gamma(\lambda+1)}{(n!)^2 s_1^{\gamma+1}s_2^{\lambda+1}} F_1\left[-n,\gamma+1,\lambda+1;\alpha+1,\beta+1;\frac{x}{s_1},\frac{y}{s_2}\right], \tag{53}$$

where F_1 is Appll's function [Srivastava and Manocha,1984].

For $\gamma=\alpha$, (52) reduces to

$$\int_0^\infty\int_0^\infty u^\alpha v^\lambda e^{-s_1u-s_2v}\, L_n^{(\alpha,\beta)}(xu,yv;k)\,du\,dv = \frac{(\beta+1)_{kn}\Gamma(\alpha+n+1)\Gamma(\lambda+1)}{(n!)^2 s_1^{\alpha+1}s_2^{\lambda+1}}$$

$$\times(1-x/s_1)^n\ {}_{k+1}F_k\left[-n,\Delta(k;\lambda+1);\Delta(k;\beta+1);\frac{(y/s_2)^k}{(1-x/s_1)}\right]. \tag{54}$$

If in (52), we let $\lambda=\beta$, we get

$$\int_0^\infty\int_0^\infty u^\gamma v^\beta e^{-s_1u-s_2v}\, L_n^{(\alpha,\beta)}(xu,yv;k)\,du\,dv = \frac{\Gamma(kn+\beta+1)(\alpha+1)_n\Gamma(\gamma+1)}{(n!)^2 s_1^{\gamma+1}s_2^{\beta+1}}$$

$$\times\left[1-(y/s_2)^k\right]^n {}_2F_1\left[-n,\gamma+1;\alpha+1;\frac{x/s_1}{1-(y/s_2)^k}\right]. \tag{55}$$

Finally, in view of the relationships (7) and (8)and using (50), equation (52) give us the following known results ([Srivastava, 1982 and Prabhakar,1970]):

$$L\{v^\lambda Z_n^\beta(yv;k):s_2\}$$

$$= \frac{(\beta+1)_{kn}\Gamma(\lambda+1)}{n!\,s_2^{\lambda+1}}\ {}_{k+1}F_k\left[-n,\frac{\lambda+1}{k},\ldots,\frac{\lambda+k}{k};\frac{\beta+1}{k},\ldots,\frac{\beta+k}{k};(y/s_2)^k\right], \tag{56}$$

$$L\{v^{\beta} Z_n^{\beta}(yv;k) : s_2\} = \frac{\Gamma(kn+\beta+1)}{n!\, s_2^{kn+\beta+1}} \left(s_2^k - y^k\right)^n \quad , \tag{57}$$

and

$$L\{u^{\alpha} L_n^{\alpha}(xu) : s_1\} = \frac{\Gamma(n+\alpha+1)}{n!\, s_1^{n+1}} \left(s_1 - x\right)^n \quad . \tag{58}$$

6. EXPANSIONS

In this section we will derive a number of expansions involving the Laguerre polynomials $L_n^{(\alpha,\beta)}(x,y;k)$,which follow:

$$\sum_{m=0}^{n} \frac{(-1)^m}{(-n)_m (kn+\beta+1)_{-km}} L_{n-m}^{(\alpha+m,\beta)}(x,y;k) \frac{z^m}{m!} = L_n^{(\alpha,\beta)}[(x-z), y : k] \quad , \tag{59}$$

$$\sum_{m=0}^{n} \frac{(-1)^m}{(-n)_m (\alpha+n+1)_{-m}} L_{n-m}^{(\alpha,\beta+km)}(x,y;k) \frac{z^{km}}{m!} = L_n^{(\alpha,\beta)}\left(x, \sqrt[k]{y^k - z^k}; k\right) \quad , \tag{60}$$

$$\sum_{m=0}^{n} \frac{(xy^k)^m}{m!} L_{n-m}^{\alpha+m}(x) \times Z_{n-m}^{\beta+km}(y;k) = L_n^{(\alpha,\beta)}(x,y;k) \quad , \tag{61}$$

$$\sum_{m=0}^{n} \frac{(-1)^m (xy^k)^m}{(-n)_m\, m!} L_{n-m}^{(\alpha+m,\beta+km)}(x,y;k) = Z_n^{\beta}(y;k) \times L_n^{(\alpha)}(x) \quad , \tag{62}$$

and

$$\sum_{m=0}^{n} \frac{(-n)_m [(n-m)!]^2 [(z^k-1)y^k]^m}{m!(n!)^2 (n+\alpha+1)_{-m}} L_{n-m}^{(\alpha,\beta+km)}(x,y;k) = L_n^{(\alpha,\beta)}(x,yz;k) \quad . \tag{63}$$

To prove (59), substitute for $L_{n-m}^{(\alpha+m,\beta)}(x,y;k)$ from (5), so we get $r \to r-m$,we obtain

$$\text{L.H.S of (59)} = \sum_{m=0}^{n} \frac{(-1)^m \Gamma(kn+\beta+1)\Gamma(\alpha+n+1)}{(-n)_m [(n-m)!]^2} \sum_{s=0}^{n-q} \sum_{r=0}^{n-q-s} \frac{(-n)_{s+r} (-r)_m x^r y^{ks} (z/x)^m}{(-1)^m (-n)_m \Gamma(ks+\beta+1)\Gamma(\alpha+r+1) m! r! s!}$$

$$= \frac{\Gamma(kn+\beta+1)\Gamma(\alpha+n+1)}{(n!)^2} \sum_{s=0}^{n} \sum_{r=0}^{n-s} \frac{(-n)_{s+r}\, x^r\, y^{ks}}{\Gamma(ks+\beta+1)\Gamma(\alpha+r+1)} \sum_{m=0}^{n} \frac{(-r)_m}{m!} (z/x)^m \, , \tag{64}$$

then we obtain the right-hand side of (59). In the same manner (60) can be proved.

To prove (61) substitute for $L_n^{(\alpha+m)}(x)$ and $Z_{n-m}^{\beta+km}(y;k)$ from (2) and (14), respectively, to get

$$\text{L.H.S.of(61)} = \sum_{m=0}^{n} \frac{(xy^k)^m \Gamma(\alpha+n+1)\Gamma(kn+\beta+1)}{m!(n!)^2} \sum_{r=0}^{n-m} \frac{(-n+m)_r x^r}{\Gamma(\alpha+r+1) r!} \sum_{s=0}^{n-m} \frac{(-n+m)_s y^{ks}}{\Gamma(ks+km+\beta+1) s!} \quad , \tag{65}$$

Now, on letting $r \to r-m$ and $s \to s-m$ in (65), and then using the transformation [Srivastava and Manocha, 1984]:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma c}{\Gamma(c-a)\Gamma(c-b)} \quad , \tag{66}$$

we obtain the right-hand side of (61).
In the same manner (62) and (63) can be proved.

Finally, on putting $z = x$ in (59) and $z = y$ in (60), we obtain the following interesting special cases:

$$\sum_{m=0}^{n} \frac{(-1)^m}{(-n)_m (kn+\beta+1)_{-km}} L_{n-m}^{(\alpha+m,\beta)}(x,y;k) \frac{x^m}{m!} = \frac{(\alpha+1)_n}{n!} Z_n^{\beta}(y;k) \quad , \tag{67}$$

and

$$\sum_{m=0}^{n} \frac{(-1)^m}{(-n)_m (\alpha+n+1)_{-m}} L_{n-m}^{(\alpha,\beta+km)}(x,y;k) \frac{y^{km}}{m!} = \frac{(\beta+1)_{kn}}{n!} L_n^{(\alpha)}(x) \quad , \tag{68}$$

respectively.

6. APPLICATIONS

An important properties of the Laguerre polynomials $L_n^{(\alpha,\beta)}(x,y;k)$ is its link to Morse potential and Pöschl-Teller potential. Indeed, a part from the harmonic oscillator potential, which provides a good first approximation for the low-lying spectrum of a diatomic molecule, the reference potential are the Morse potential (which describes reasonably well the high-lying spectrum) and the Pöschl-Teller (which describes better some specific degree of freedom of linear molecules, such as the flexion modes) [Sánchez-Ruiz, 2003]. In this section we will see how some of the results of the previous sections can be exploited to derive expansion formulas for the wavefunction of Morse potential and of Pöschl-Teller potential, which are specially useful in situations when the parameters and variables take on particular values.

(a) *Expansion of Morse wavefunction in integrals of elementary functions*

The so-called Morse potential is

$$V(w) = V_0\,(e^{-2\gamma w} - 2e^{-\gamma w}), \tag{69}$$

where V_0 and γ are cnstants. The exact closed form of the normalized eigenfunctions is ([Nieto et.al., 1979; Pérez-Bernal et.al., 2000])

$$\psi_n(w) = N(n)\, x^{\lambda-n-1/2}\, e^{-x/2}\, L_n^{(2\lambda-2n-1)}(x) \ , \tag{70}$$

for $0 \le n \le [\lambda - 1/2]$ (square brackets denote integer part of the expression within), where

$$\lambda = \left(\frac{2\sigma V_0}{\hbar^2\gamma^2}\right) , \qquad x = 2\lambda e^{-\gamma w}, \qquad N(n) = \left(\frac{\gamma(2\lambda-2n-1)n!}{\Gamma(2\lambda-n)}\right)^{1/2} , \tag{71}$$

σ is the mass of particle and $L_n^{(2\lambda-2n-1)}(u)$ is the varying Laguerre polynomials define by (equation (2)):

$$L_n^{(2\lambda-2n-1)}(u) = \frac{(2\lambda-2n)_n}{n!}\, {}_1F_1\left[\begin{matrix} -n; \\ 2\lambda-2n; \end{matrix}\ u\right] \ . \tag{72}$$

For the involved Laguerre polynomials in (72), the integral representation (27) yields the connection formula

$$L_n^{(2\lambda-2n-1)}(x) = \frac{\Gamma(2\lambda-n)}{n!\,(2\pi i)^2} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} \frac{(t - xt/u)^n}{t^{n+1} u^{2\lambda-n}}\, e^{t+u}\, dt\, du \quad , \tag{73}$$

and therefore the corresponding relationship for the Morse wavefunction is

$$\psi_n(w) = \frac{N(n)\, x^{\lambda-n-1/2}\, e^{-x/2}}{n!\,(2\pi i)^2} \int_{-\infty}^{(0+)} \int_{-\infty}^{(0+)} \frac{(t - xt/u)^n}{t^{n+1}\, u^{2\lambda-n}}\, e^{t+u}\, dt\, du \ , \qquad (74)$$

which is an expansion of Morse wavefunction in form of Hankel's transform.

(b) *Expansion of Pöschl-Teller wavefunction in integral involving Laguerre polynomials*

Let us consider Pöschl-Teller potential

$$V(w) = -\frac{W_0}{\cosh^2 kw}\ ,\ \ W_0 > 0 \ \ ,$$

which is often used as a realistic model for molecular potentials. Its eigenfunctions are essentilly varying Gegenbauer polynomials, since their explicit form is [Nikiforov and Uvarvo, 1988; Pérez-Bernal et.al., 2000]

$$\phi_n(w) = C_n\,(1-x^2)^{\sigma/2}\ P_n^{(\sigma,\sigma)}(x) \ , \qquad (75)$$

where $P_n^{(\sigma,\sigma)}$ is the Gegenbauer polynomials defined by [Rainville, 1960]:

$$P_n^{(\sigma,\sigma)}(x) = \frac{(\sigma+1)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n, n+2\sigma+1; & \\ & \dfrac{1-x}{2} \\ \sigma+1; & \end{matrix}\right] , \qquad (76)$$

for $n \le \left(\sqrt{s^2+1/4} - 1/2\right)$, $\sigma = \sigma_n = -n + \sqrt{s^2+1/4} - 1/2$, $x = \tanh kw$,
$s^2 = \dfrac{2\mu W_0}{\hbar^2 k^2}$ and C_n is the normalization constant.

For the involved Gegenbauer polynomials in (76) the Laplace transform (55) yields the connection formula

$$P_n^{(\sigma,\sigma)}(x) = \frac{1}{\Gamma(2\sigma+n+1)} \int_0^\infty u^{2\sigma+n}\, e^{-u}\, L_n^{(\sigma)}\left(\frac{(1-x)}{2}u\right) du \ , \qquad (77)$$

and therefore the corresponding relationship for Pöschl-Teller wavefunction is

$$\phi_n(w) = \frac{C_n\,(1-x^2)^{\sigma/2}}{\Gamma(2\sigma+n+1)} \int_0^\infty u^{2\sigma+n}\, e^{-u}\, L_n^{(\sigma)}\left(\frac{(1-x)}{2}u\right) du \ . \qquad (78)$$

(c) *The product of two Morse wavefunctions*

Now, we will compute the connection formula relating the product of two wavefunctions of two Morse potential with different parameters.
Let equations (69), (70) and (71) be the first Morse potential. For the second Morse potentail,

$$\tilde{V}(w) = \tilde{V}_0(e^{-2\tilde{\gamma} w} - 2e^{-\tilde{\gamma} w}) ,$$

the eigenfunctions $\tilde{\psi}_n(w)$ are given by equations (70) and (71) with $V_0, \gamma, \lambda, x, N(n)$ replaced by $\tilde{V}_0, \tilde{\gamma}, \tilde{\lambda}, \tilde{x}, \tilde{N}(n)$, respectively.

For $k = 1$, formula (62) reduces to

$$\sum_{m=0}^{n} \frac{(-1)^m (xy\ )^m}{(-n)_m m!} L_{n-m}^{(\alpha+m,\beta+m)}(x,y) = L_n^{(\beta)}(y) \times L_n^{(\alpha)}(x) \ . \tag{79}$$

For the involved Laguerre polynomials, equation (79) give us

$$L_n^{(2\tilde{\lambda}-2n-1)}(y) \times L_n^{(2\lambda-2n-1)}(x) = \sum_{m=0}^{n} \frac{(-1)^m (xy)^m}{(-n)_m m!} L_{n-m}^{(2\lambda-2n+m-1, 2\tilde{\lambda}-2n+m-1)}(x,y) \,, \tag{80}$$

and therefore the corresponding relationship for the product of two Morse wavefunctions is

$$\tilde{\psi}_n(w) \times \psi_n(w) = \tilde{N}(n)\, N(n)\, x^{\lambda-n-1/2}\ y^{\tilde{\lambda}-n-1/2}\ e^{-(x+y)/2}$$
$$\times \sum_{m=0}^{n} \frac{(-1)^m (xy)^m}{(-n)_m m!} L_{n-m}^{(2\lambda-2n+m-1, 2\tilde{\lambda}-2n+m-1)}(x,y) \,. \tag{81}$$

7. CONCLUSIONS

Section 2 introduced a new class of Laguerre polynomials denoted by $L_n^{(\alpha,\beta)}(x,y;k)$ as the natural solution of partial differential equation of the type given by equation (47).
It has been shown that they are unification and generalization of a number of known Laguerre polynomials. Further, It has been observed that the use of Laguerre polynomials $L_n^{(\alpha,\beta)}(x,y;k)$ is particularly useful to derive new properties for already known polynomials, which are a special cases of $L_n^{(\alpha,\beta)}(x,y;k)$.
Finally, the same point of view of the previous sections can be used to generalize other special functions of mathematical physics and state new properties,like, generating functions, integral representations, differential equation, recurrence relations, Laplace transform, series expansions and some applications, for these special functions, which play an important role in physics and engineering.

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