N- Fractional Differintegration of The Multiple Hurwitz Zeta Functions

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Abstract

In this paper ,by using N- Fractional Differintegration , we have made a considerable results on the multiple Hurwitz Zeta functions $\zeta(z,a)$ and $\zeta(-z,a)$.

Keywords: Generalized Zeta function , multiple Hurwitz Zeta function.
N- Fractional Differintegration , N- fractional Differintegration ,
N- partial and fractional Differintegrals.

1. Introduction and definitions

By applying N- fractional calculus, Nishomoto (see [1] and [2]) established new theorems involving the fractional calculus of the generalized Zeta functions $\zeta(z,a)$ and $\zeta(-z,a)$. Also Nishimoto (see [3]) established the N- partial and fractional Differintegrals calculus of the generalized Zeta function.

In [2] Nishomoto gave the following definitions:

Definition 1: Let

$D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$

(Here D contains the point over the curve C).

Moreover , let $f = f(z)$ be a regular function in $D\ (z \in D)$.

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,d\zeta \ , \quad (\nu \notin \mathbb{Z}^-), \quad (1.1)$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \ , \quad (m \in \mathbb{Z}^+) \quad (1.2)$$

where $$-\pi \le \arg\left(\zeta - z\right) \le \pi \,, \quad \text{for } C_-$$
$$0 \le \arg\left(\zeta - z\right) \le 2\pi \,, \quad \text{for } C_+$$

Γ is the Gamma function, then $\left(f\right)_\nu$ is the fractional Differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order - ν for $\nu < 0$, with respect to z , of the function f.

Definition 2: The fractional calculus operator (Nishomoto's operator) is defined as

$$N^{\nu} = \frac{\Gamma\left(\nu+1\right)}{2\pi i}\int_C \frac{d\zeta}{\left(\zeta - z\right)^{\nu+1}}\, d\zeta \quad , \quad \left(\nu \notin \mathbb{Z}^-\right), \qquad (1.3)$$

with

$$N^{-m} = \lim_{\nu \to -m} N^{\nu} \quad , \quad \left(m \in \mathbb{Z}^+\right) \qquad (1.4)$$

In [4] the following definition due to Junesang Choi and Srivastava is given:

Definition 3: The multiple Hurwitz Zeta function is defined by

$$\zeta_n\left(z\,,a\right) = \sum_{k_1,\ldots,k_n=0}^{\infty} \left(a + k_1 + \ldots + k_n\right)^{-z} \,, \qquad (1.5)$$

where

$$z = \sigma + it \;,\; \left(\sigma\,, t \in \mathbb{R} \;,\; \operatorname{Re} z = \sigma > n \;,\; a \notin \mathbb{Z}_0\right),$$

and the integral representation of $\zeta_n\left(z\,,a\right)$ is

$$\zeta_n\left(z\,,a\right) = \frac{1}{\Gamma\left(z\right)}\int_0^{\infty} \frac{t^{z-1}e^{-at}}{\left(1-e^{-t}\right)^n}\, dt \quad , \quad \operatorname{Re}\left(z\right) > n \,,\, n \in \mathbb{N} \qquad (1.6)$$

In (1.5) replacing z by $-z$ we get

Definition 4:

$$\zeta_n\left(-z\,,a\right) = \sum_{k_1,\ldots,k_n=0}^{\infty} \left(a + k_1 + \ldots + k_n\right)^{z} \quad , \qquad (1.7)$$

$$\mathrm{Re}(z) < -n \qquad , \; a \notin \mathbb{Z}_0$$

Finally, let us recall some useful results :

Theorem 1.1: $\left(e^{az}\right)_\nu = a^\nu e^{az}$, for $a \neq 0$, $(z , \nu \in \mathbb{C})$, (cf. [2])

Theorem 1.2 : (i) $\left(\cos az\right)_\nu = a^\nu \cos\left(az + \frac{\pi}{2}\nu\right)$,

for $a \neq 0$, $(z , \nu \in \mathbb{C})$, (cf. [1])

(ii) $\left(\sin az\right)_\nu = a^\nu \sin\left(az + \frac{\pi}{2}\nu\right)$,

for $a \neq 0$, $(z , \nu \in \mathbb{C})$, (cf. [1])

2: Main results

Theorem 2.1 : We have

$$(i) \quad \zeta_n^{(\alpha)}(z,a) = e^{i\pi\alpha} \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\alpha} (a+k_1+\dots+k_n)^{-z} \;,$$

$$(ii) \quad \zeta_n^{(\alpha)}(-z,a) = \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\alpha} (a+k_1+\dots+k_n)^{-z} \;,$$

for $\alpha \in \mathbb{R}$,

Proof of (i): From (1.5), we have

$$\zeta_n(z,a) = \sum_{k_1,\dots,k_n=0}^{\infty} (a+k_1+\dots+k_n)^{-z} \;, \quad \mathrm{Re}(z) > n \,, \quad a \notin \mathbb{Z}_0^-$$

$$= \sum_{k_1,\dots,k_n=0}^{\infty} e^{\log(a+k_1+\dots+k_n)^{-z}} = \sum_{k_1,\dots,k_n=0}^{\infty} e^{z\log(a+k_1+\dots+k_n)^{-1}} \;, \qquad (a)$$

Now applying Nishomoto's operator N^α defined by (1.3) to both sides of (a) , and using Theorem 1.1, we get

$$\zeta_n^{(\alpha)}(z,a) = \sum_{k_1,\dots,k_n=0}^{\infty} \left[e^{z \log(a+k_1+\dots+k_n)^{-1}} \right]_\alpha$$

$$= \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)^{-1} \right]^\alpha e^{z \log(a+k_1+\dots+k_n)^{-1}}$$

$$= (-1)^\alpha \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n) \right]^\alpha (a+k_1+\dots+k_n)^{-z}$$

$$= e^{i\pi\alpha} \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n) \right]^\alpha (a+k_1+\dots+k_n)^{-z}$$

which ends the proof.

Proof of (ii): Operating the operator N^{α} on both sides of (1.7), hence the proof is the same as in (i).

Note 1: When $\alpha > 0$, we have $\zeta_n^{(\alpha)}(z,a) = \dfrac{d^\alpha \zeta_n(z,a)}{dz^\alpha}$.

Theorem 2.2 : We have

(i) $\left[\zeta_n^{(\alpha)}(z,a)\right]_\beta = \left[\zeta_n^{(\beta)}(z,a)\right]_\alpha = \zeta_n^{(\alpha+\beta)}(z,a)$, $\mathrm{Re}(z) > n$,

(ii) $\left[\zeta_n^{(\alpha)}(-z,a)\right]_\beta = \left[\zeta_n^{(\beta)}(-z,a)\right]_\alpha = \zeta_n^{(\alpha+\beta)}(-z,a)$, $\mathrm{Re}(z) < -n$,

for $\alpha \in \mathbb{R}$,

Proof: This Theorem can be easily proved by using Theorem 1.

Theorem 2.3 : (N- partial and fractional Differintegrals of $\zeta_n(z,a)$)

We have :

$$(i)\quad \zeta_n^{(\alpha(x))}(z,a)=\zeta_n^{(\alpha(x))}(x+iy,a)$$
$$=\sum_{k_1,\dots,k_n=0}^{\infty}\left[\log\left(a+k_1+\dots+k_n\right)^{-1}\right]^{\alpha}\left(a+k_1+\dots+k_n\right)^{-z},$$

$$(ii)\quad \zeta_n^{(\beta(y))}(z,a)=\zeta_n^{(\beta(y))}(x+iy,a)$$
$$=\sum_{k_1,\dots,k_n=0}^{\infty}\left[i\log\left(a+k_1+\dots+k_n\right)^{-1}\right]^{\beta}\left(a+k_1+\dots+k_n\right)^{-z},$$

for $\alpha,\beta\in\mathbb{R}$, $\mathrm{Re}(z)=x>n$

Proof of (i): We have

$$\zeta_n^{(\alpha(x))}(z,a)=\zeta_n^{(\alpha(x))}(x+iy,a)=\sum_{k_1,\dots,k_n=0}^{\infty}\left[\left(a+k_1+\dots+k_n\right)^{-(x+iy)}\right]_{\alpha(x)}$$

$$=\sum_{k_1,\dots,k_n=0}^{\infty}\left[\left(a+k_1+\dots+k_n\right)^{-x}\left(a+k_1+\dots+k_n\right)^{-iy}\right]_{\alpha(x)}$$

$$=\sum_{k_1,\dots,k_n=0}^{\infty}\left[e^{x\log\left(a+k_1+\dots+k_n\right)^{-1}}e^{iy\log\left(a+k_1+\dots+k_n\right)^{-1}}\right]_{\alpha(x)}$$

$$=\sum_{k_1,\dots,k_n=0}^{\infty}\left[e^{x\log\left(a+k_1+\dots+k_n\right)^{-1}}\right]_{\alpha(x)}e^{iy\log\left(a+k_1+\dots+k_n\right)^{-1}}.$$

Now using Theorem (1.1) and by easy calculations we get the desired result.

Proof of (ii): We have

$$\zeta_n^{(\beta(y))}(z,a)=\zeta_n^{(\beta(y))}(x+iy,a)=\sum_{k_1,\dots,k_n=0}^{\infty}\left[\left(a+k_1+\dots+k_n\right)^{-(x+iy)}\right]_{\beta(y)}$$

$$=\sum_{k_1,\dots,k_n=0}^{\infty}\left[\left(a+k_1+\dots+k_n\right)^{-x}\left(a+k_1+\dots+k_n\right)^{-iy}\right]_{\beta(y)}$$

$$= \sum_{k_1,\dots,k_n=0}^{\infty} \left[e^{\log(a+k_1+\dots+k_n)^{-x1}} e^{iy\log(a+k_1+\dots+k_n)^{-1}} \right]_{\beta(y)}$$

$$= \sum_{k_1,\dots,k_n=0}^{\infty} \left[e^{iy\log(a+k_1+\dots+k_n)^{-1}} \right]_{\beta(y)} e^{\log(a+k_1+\dots+k_n)^{-x1}}$$

$$= \sum_{k_1,\dots,k_n=0}^{\infty} \left[i\log(a+k_1+\dots+k_n)^{-1} \right] e^{iy\log(a+k_1+\dots+k_2)^{-1}} e^{\log(a+k_1+\dots+k_n)^{-x1}},$$

which gives the desired result.

Theorem 2.4 : Let $\operatorname{Re}\zeta_n(z,a) = u(x,y) = u$ and $\operatorname{Im}\zeta_n(z,a) = v(x,y) = v$, we have

$$(i)\ u_{\alpha(x)} = e^{i\pi\alpha} \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\alpha} (a+k_1+\dots+k_n)^{-x} \times\cos\left[y\log(a+k_1+\dots+k_n)\right],$$

$$(ii)\ u_{\beta(y)} = e^{i\pi\beta} \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\beta} (a+k_1+\dots+k_n)^{-x} \times\cos\left[y\log(a+k_1+\dots+k_n)^{-1} + \frac{\pi}{2}\beta\right],$$

$$(iii)\ v_{\alpha(x)} = e^{i\pi(\alpha+1)} \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\alpha} (a+k_1+\dots+k_n)^{-x} \times\sin\left[y\log(a+k_1+\dots+k_n)\right],$$

$$(iv)\ v_{\beta(y)} = e^{i\pi\beta} \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\beta} (a+k_1+\dots+k_n)^{-x} \times\sin\left[y\log(a+k_1+\dots+k_n)^{-1} + \frac{\pi}{2}\beta\right],$$

for $\alpha, \beta \in \mathbb{R}$, $\operatorname{Re}(z) = x > n$,

Proof : We have

$$\zeta_n(z,a)=\sum_{k_1,\dots,k_n=0}^{\infty}(a+k_1+\dots+k_n)^{-z}=\sum_{k_1,\dots,k_n=0}^{\infty}(a+k_1+\dots+k_n)^{-(x+iy)}$$

$$=\sum_{k_1,\dots,k_n=0}^{\infty}e^{x\log(a+k_1+\dots+k_n)^{-1}}e^{iy\log(a+k_1+\dots+k_n)^{-1}}$$

$$=\sum_{k_1,\dots,k_n=0}^{\infty}e^{x\log(a+k_1+\dots+k_n)^{-1}}$$

$$\times\left\{\cos\left(y\log(a+k_1+\dots+k_n)^{-1}\right)+i\sin\left(y\log(a+k_1+\dots+k_n)^{-1}\right)\right\}.$$

Hence

$$u=\operatorname{Re}\zeta_n(z,a)=\sum_{k_1,\dots,k_n=0}^{\infty}e^{x\log(a+k_1+\dots+k_n)^{-1}}$$

$$\times\cos\left(y\log(a+k_1+\dots+k_n)^{-1}\right),\qquad(b)$$

$$v=\operatorname{Im}\zeta_n(z,a)=\sum_{k_1,\dots,k_n=0}^{\infty}e^{x\log(a+k_1+\dots+k_n)^{-1}}$$

$$\times\sin\left(y\log(a+k_1+\dots+k_n)^{-1}\right),\qquad(c)$$

using Theorem (1.2) ,we can get (i) ,(ii) from (b), and (iii), (iv) from (c) respectively.

Also we can get N- partial and fractional differintegral of $\zeta_n(-z,a)$ in the following theorem

Theorem 2.5 : Let $\operatorname{Re}\zeta_n(-z,a)=u(x,y)=u$ and $\operatorname{Im}\zeta_n(-z,a)=v(x,y)=v$, we have

$$(i)\quad u_{\alpha(x)}=\sum_{k_1,\dots,k_n=0}^{\infty}\left[\log(a+k_1+\dots+k_n)\right]^{\alpha}(a+k_1+\dots+k_n)^{x}$$

$$\times\cos\left[y\log(a+k_1+\dots+k_n)\right],$$

$$(ii)\ u_{\beta(y)} = \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\beta} (a+k_1+\dots+k_n)^{x}$$
$$\times \cos\left[y \log(a+k_1+\dots+k_n)^{-1} + \frac{\pi}{2}\beta\right],$$

$$(iii)\ v_{\alpha(x)} = \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\alpha} (a+k_1+\dots+k_n)^{x}$$
$$\times \sin\left[y \log(a+k_1+\dots+k_n)\right],$$

$$(iv)\ v_{\beta(y)} = \sum_{k_1,\dots,k_n=0}^{\infty} \left[\log(a+k_1+\dots+k_n)\right]^{\beta} (a+k_1+\dots+k_n)^{x}$$
$$\times \sin\left[y \log(a+k_1+\dots+k_n) + \frac{\pi}{2}\beta\right],$$

for $\alpha, \beta \in \mathbb{R}$, $\mathrm{Re}(z) = x < -n$,

Proof : The proof of this Theorem is in the same manner that of Theorem 2.4.

Nishomoto in [3] gave two theorems concerning N- partial and fractional differintegral , and N- partial differintegral equations for $\zeta(z,a)$, here we have applying those results as an extension for multiple Hurwitz Zeta function $\zeta_n(z,a)$.

Theorem 2.6 :

$$\zeta_n^{\alpha(x)}(z,a) - (-i)^{\alpha}\, \zeta_n^{\alpha(y)}(z,a) = 0 \quad , \quad \mathrm{Re}(z) = x > n\ ,\ \alpha \in \mathbb{R}.$$

Proof : We have from Theorem 2.3 (ii)

$$\zeta_n^{(\alpha(y))}(z,a) = \sum_{k_1,\dots,k_n=0}^{\infty} i^{\alpha} \left[\log(a+k_1+\dots+k_n)^{-1}\right]^{\alpha} (a+k_1+\dots+k_n)^{-z}$$

on the other hand we have from Theorem 2.3 (i)

$$i^{\alpha}\zeta_n^{(\alpha(x))}(z,a) = i^{\alpha}\sum_{k_1,\ldots,k_n=0}^{\infty}\left[\log(a+k_1+\ldots+k_n)^{-1}\right]^{\alpha}(a+k_1+\ldots+k_n)^{-z}$$

now $i^{\alpha}\zeta_n^{\alpha(x)}(z,a) - \zeta_n^{\alpha(y)}(z,a) = 0$, multiplying both sides of this equation by $(-i)^{\alpha}$ we get the theorem.

Theorem 2.7:

$$\zeta_n^{m(x)}(z,a) - (-i)^n\,\zeta_n^{m(y)}(z,a) = 0 \quad , \quad \operatorname{Re}(z) = x > n \ , \ m \in \mathbb{Z}.$$

Proof : The proof of this theorem can be easily found from Theorem 2.5 by setting $\alpha = m$.

Setting $\alpha = m \in \mathbb{Z}^+$ in Theorem (2.6), we get the following result

Corollary 1:

$$\frac{\partial^m \zeta_n(z,a)}{d\,x^m} - (-i)^m \frac{\partial^m \zeta_n(z,a)}{d\,y^m} = 0 \quad , \quad \operatorname{Re}(z) = x > n .$$

Also setting $\alpha = m \in \mathbb{Z}^-$ in theorem (3.6), we get the following result

Corollary 2:

$$\int \zeta_n(z,a)(dx)^{-m} - (-i)^m \int \zeta_n(z,a)(dy)^{-m} = 0 .$$

Now, if we Put $n = 1$ in Theorem (2.1) and (2.2), we get the Theorems 2 and 3 given by Nishomoto in [3] related to N- ordinary fractional calculus of $\zeta(z,a)$ and $\zeta(-z,a)$.

Also put $n = 1$ in Theorems (2.3) , (2.4), and (2.5), we get Theorems 1 and 2 given by Nishomoto in [3] .

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التفاضل والتكامل ألكسري لنشيموتو لدالة زيتا هورتز المضاعفة

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الملخص
في هذا البحث وباستخدام الاشتقاق والتكامل ألكسري لنشيموتو [3]، قمنا بتطوير عدد من النتائج على دالتي زيتا هورتز المضاعفة $\zeta_n\left(z\ ,a\right)$ و $\zeta_n\left(-z\ ,a\right)$ والحصول على نتائج معتبرة.

الكلمات المفتاحية: دالة زيتا العامة . دالة زيتا هورتز المضاعفة . الاشتقاق ألكسري لنشوموتو . التكامل ألكسري لنشوموتو . التفاضل والتكامل ألكسري الجزئي.